Errata to *Tikhonov and Landweber convergence rates: characterization by interpolation spaces*

April 24, 2026

Abstract

This note records errata for the paper *Tikhonov and Landweber convergence rates: characterization by interpolation spaces*. Equation, section, lemma, and proposition numbers refer to the arXiv version `https://arxiv.org/abs/1503.05742v4`. The errata list below states the corrections succinctly, and the subsequent notes give detailed justifications for selected items.

Errata list

(E1) In equation (6), the displayed value

$$\alpha = t^{1/k}\left(\frac{1-\theta}{\theta}\right)^{1-1/(2k)}$$

is meaningful only for $0 < \theta < 1$. The endpoint cases $\theta = 0, 1$ should be treated separately, or obtained by a limiting argument.

(E2) In Section 2.3, the sentence claiming that the distance function d has compact support if and only if $x \in X_1$ is not valid for an arbitrary continuous embedding $X_1 \subset X$. It should be deleted, or stated under an additional closedness assumption on the embedded balls of X_1 in X.

(E3) The identity (9)–(10) is false as stated. The quantity

$$E := \sup_{r>0} \inf_{\|x_1\|_1 \le r} \|x - x_1\|_0^{1-\theta} \|x_1\|_1^{\theta}$$

does not equal $N_\theta^{-1}\|x\|_{\theta:1}$ in general. The valid identity is

$$N_\theta^{-1}\|x\|_{\theta:1} = D, \qquad D := \sup_{r>0} d(r)^{1-\theta} r^{\theta}.$$

Consequently, the proof of (9), in particular step c), should be replaced.

(E4) In Lemma 2.1, the quantifier in inequality (13) should read

$$\forall\, \gamma > \nu$$

instead of $\forall\, \gamma > 0$.

(E5) In the proof of Proposition 2.2, inequality (26a), after the substitution $s := t^{1/\gamma}$ the term

$$s^2 \int_{[s,\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda)$$

should be

$$s^{2\gamma} \int_{[s,\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda).$$

Equivalently, one may keep the term as $t^2 \int_{[s,\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda)$ until substituting $s = t^{1/\gamma}$.

(E6) In the proof of Proposition 2.2, inequality (26b), the displayed choice

$$t = s^{\gamma}\sqrt{\frac{\gamma-\nu}{\nu}}$$

is meaningful only for $0 < \nu < \gamma$. The endpoint case $\nu = 0$ should be handled separately.

(E7) Definitions and statements involving the quotient ν/γ require $\gamma > 0$. In particular, the definition of $X_{\nu:\gamma}$ and Proposition 2.4 should be read with this restriction whenever ν/γ appears.

(E8) The lower-bound part of the proof of Proposition 2.4 should not invoke the false E-identity in (9)–(10). It should use only the corrected distance-function identity

$$N_{\nu}^{-1}\|x\|_{\nu:1} = \sup_{r>0} d(r)^{1-\nu} r^{\nu},$$

or be replaced by a direct Hilbert-space duality argument.

(E9) Equations (55)–(56) contain an incorrect minimization problem and an inconsistent parameter scaling. To keep (55) with the factor $\alpha^{-\nu}$, equation (56) should read

$$x_{\alpha:\gamma} := \arg\min_{x\in X} \left\{ \|(T^*T)^{\gamma/2}(x^{\dagger} - x)\|^2 + \alpha^{\gamma}\|x - x_0\|^2 \right\}.$$

Equivalently, if a parameter β is placed in front of $\|x-x_0\|^2$, then the middle term in (55) should be

$$\sup_{\beta>0} \beta^{-\nu/\gamma}\|x^{\dagger} - x_{\beta:\gamma}\|.$$

(E10) In the proof of Proposition 3.2, the appeal to equation (6) does not literally cover the endpoint cases $\nu = 0$ and $\nu = k$. The proposition is correct, but these two cases should be justified separately.

(E11) The residual representation (64) is missing the factor σ. It should read

$$y^{\dagger} - Tx_k = (I - \sigma TT^*)^k (y^{\dagger} - Tx_0), \qquad y^{\delta} - Tx_k^{\delta} = (I - \sigma TT^*)^k (y^{\delta} - Tx_0).$$

(E12) For general step size σ, the noisy Landweber splitting (65) should contain a factor $\sqrt{\sigma}$. Namely,

$$\|x^{\dagger} - x_k^{\delta}\| \le \|x^{\dagger} - x_k\| + \delta\sqrt{\sigma k}.$$

The printed estimate is valid after the normalization $\sigma = 1$, or under an additional harmless adjustment of constants when σ is fixed.

(E13) In equation (67), the factor $(1 + k/\nu)^{\nu}$ requires $\nu > 0$. Thus the surrounding statement should not allow $\nu = 0$ without a separate convention.

(E14) In the proof of Proposition 3.4, equation (74) is preceded by the claim that ε_k maximizes

$$\lambda^{2(r+\nu)}(1 - \sigma\lambda)^{2k}$$

over $\lambda \ge 0$. The correct maximization domain is the spectral interval

$$0 \le \lambda \le 1/\sigma.$$

(E15) In the paragraph following equation (87), the sufficient condition for the discrepancy principle has the inequality reversed. To ensure

$$\|y^{\dagger} - Tx_k\| \le \delta(\tau - 1),$$

one needs

$$c_2\varepsilon_k^{\nu+1/2}\|\!|x^{\dagger} - x_0|\!\|_{\nu} \le \delta(\tau - 1),$$

not the reverse inequality.

(E16) The asymptotic estimate displayed in equation (84) after Proposition 3.4 is written in the normalization $\sigma = 1$. For general σ, the factor

$$\varepsilon_k^{r+\nu} = \sigma^{-(r+\nu)}\left(\frac{r+\nu}{k+r+\nu}\right)^{r+\nu}$$

contributes the corresponding power of σ.

Erratum E1: the endpoint formulation of equation (6)

In the sentence preceding equation (6), the range $0 \le \theta \le 1$ should be replaced by $0 < \theta < 1$ if the displayed value of α is used. Indeed,

$$\alpha = t^{1/k}\left(\frac{1-\theta}{\theta}\right)^{1-1/(2k)}$$

is not defined at the endpoints $\theta = 0$ and $\theta = 1$. The endpoint cases should instead be treated separately, or obtained by an explicit limiting argument.

For completeness, let us recall the verification of equation (6) in the corrected range $0 < \theta < 1$. Put

$$p := 2k \ge 1.$$

With the displayed choice of α, one has

$$\alpha^p = t^2\left(\frac{1-\theta}{\theta}\right)^{p-1}, \qquad \text{or equivalently} \qquad t^2 = \alpha^p\left(\frac{\theta}{1-\theta}\right)^{p-1}.$$

Using $N_\theta^{-2} = \theta^\theta(1-\theta)^{1-\theta}$, the left hand side of equation (6) becomes

$$N_\theta^{2(1-p)}\,\frac{t^{2(1-\theta)}}{t^2+\lambda^p} = \frac{\theta^{p-1}\alpha^{p(1-\theta)}}{\left(\frac{\theta}{1-\theta}\right)^{p-1}\alpha^p + \lambda^p}.$$

Thus equation (6) is equivalent to

$$(\alpha+\lambda)^p \le (1-\theta)^{1-p}\alpha^p + \theta^{1-p}\lambda^p.$$

This last estimate follows from convexity of $s \mapsto s^p$:

$$(\alpha+\lambda)^p = \left((1-\theta)\frac{\alpha}{1-\theta} + \theta\frac{\lambda}{\theta}\right)^p \le (1-\theta)^{1-p}\alpha^p + \theta^{1-p}\lambda^p.$$

This proves the corrected form of equation (6).

At the endpoints the displayed value of α should not be used. The relevant limiting scalar estimates are elementary. For $\theta = 0$, $N_0 = 1$ and

$$\frac{t^2}{t^2+\lambda^{2k}} \le 1 = \lim_{\alpha\to\infty}\left(\frac{\alpha}{\alpha+\lambda}\right)^{2k}.$$

For $\theta = 1$, $N_1 = 1$ and, for $\lambda > 0$,

$$\frac{1}{t^2+\lambda^{2k}} \le \lambda^{-2k} = \lim_{\alpha\downarrow 0}\left(\frac{1}{\alpha+\lambda}\right)^{2k}.$$

The case $\lambda = 0$ is interpreted in the extended sense in the last inequality; in the spectral applications of the paper, mass at $\lambda = 0$ is absent because T is assumed injective. Hence any proof which uses equation (6) at $\theta = 0$ or $\theta = 1$ should replace that invocation by the corresponding endpoint argument.

Erratum E2: compact support of the distance function

In Section 2.3 ("Distance function"), the sentence "It is clear that d has compact support if and only if $x \in X_1$" is not valid for a general Banach space X_1 continuously embedded in X. The correct statement is

$$d \text{ has compact support} \quad\Longleftrightarrow\quad x \in \bigcup_{R>0} \overline{\{x_1 \in X_1 : \|x_1\|_1 \le R\}}^{X}.$$

In particular, $x \in X_1$ implies that d has compact support, but the converse requires an additional assumption, for instance that the closed balls of X_1 are closed in X.

Here is a counterexample to the converse as stated. Let $X = \ell^2(\mathbb{N}_0)$ with orthonormal basis $u_0, u_1, u_2, \ldots$, and let $Z = \ell^1(\mathbb{N})$. Define a bounded injective operator $J : Z \to X$ by

$$Ja := \left(\sum_{n\geq 1} a_n\right) u_0 + \sum_{n\geq 1} \frac{a_n}{n} u_n.$$

Let $X_1 := JZ \subset X$ and equip X_1 with the norm

$$\|Ja\|_1 := \|a\|_{\ell^1}.$$

Then X_1 is a Banach space continuously embedded in X. Indeed,

$$\|Ja\|_X^2 = \left|\sum_{n\geq 1} a_n\right|^2 + \sum_{n\geq 1} \frac{|a_n|^2}{n^2} \leq 2\|a\|_{\ell^1}^2.$$

Take $x := u_0 \in X$. Then $x \notin X_1$: if $Ja = u_0$, then comparison of the u_n-coordinates for $n \geq 1$ gives $a_n = 0$ for all n, and then the u_0-coordinate is also zero, a contradiction.

On the other hand, for the unit vectors $e_n \in \ell^1(\mathbb{N})$ one has

$$Je_n = u_0 + \frac{1}{n} u_n, \qquad \|Je_n\|_1 = 1, \qquad \|Je_n - u_0\|_X = \frac{1}{n}.$$

Consequently, for the distance function associated with this x,

$$d(1) \leq \inf_{n\geq 1} \|Je_n - u_0\|_X = 0.$$

Thus $d(r) = 0$ for all $r \geq 1$, so d has compact support, while $x \notin X_1$. This disproves the printed equivalence in the stated generality.

The printed sentence becomes correct under the additional hypothesis that each ball

$$\{x_1 \in X_1 : \|x_1\|_1 \leq R\}$$

is closed in X. Under that hypothesis, $d(R) = 0$ implies that x lies in such a closed ball, hence $x \in X_1$. This extra hypothesis holds, for example, when X_1 is reflexive and the embedding $X_1 \hookrightarrow X$ is continuous.

Erratum E3: the distance-function identity (9)–(10)

The identity (9) is false as stated because the quantity E in (10) is not the right distance quantity. Namely, for any $r > 0$ the vector $x_1 = 0$ is admissible in the infimum defining E, and therefore

$$\inf_{\|x_1\|_1 \leq r} \|x - x_1\|_0^{1-\theta} \|x_1\|_1^{\theta} = 0.$$

Thus $E = 0$ for every x. In general this cannot equal $N_\theta^{-1}\|x\|_{\theta:1}$. For instance, if $X = X_1 = \mathbb{R}$, both with the usual norm, and $x = 1$, then

$$d(r) = \max\{1 - r, 0\}, \qquad D = \sup_{0<r<1} (1-r)^{1-\theta} r^{\theta} = (1-\theta)^{1-\theta}\theta^{\theta} > 0,$$

whereas the printed quantity E is zero.

The correct statement is

$$N_\theta^{-1}\|x\|_{\theta:1} = D, \qquad D := \sup_{r>0} d(r)^{1-\theta} r^{\theta}. \tag{1}$$

Consequently, step c) in the proof of (9) should be deleted. The passage from an infimum over $\|x_1\|_1 \leq r$ to an infimum over $\|x_1\|_1 = r$ is not valid for the printed expression E.

We include a proof of the corrected identity (1). Put

$$F := \|x\|_{\theta:1}, \qquad h(r) := d(r)^2.$$

From the definitions of K_t and d, one has

$$K_t(x)^2 = \inf_{r\geq 0}\{h(r) + t^2 r^2\}. \tag{2}$$

Indeed, the inequality "$\geq$" follows by taking $r = \|x_1\|_1$ in the definition of K_t, while the reverse inequality follows by approximating the infimum in the definition of $d(r)$.

First suppose that $D < \infty$. Then, for every $r > 0$,

$$d(r) \leq D^{1/(1-\theta)} r^{-\theta/(1-\theta)}.$$

Using this in (2) and minimizing over $r > 0$ gives

$$K_t(x)^2 \leq \inf_{r>0}\left\{D^{2/(1-\theta)} r^{-2\theta/(1-\theta)} + t^2 r^2\right\} = N_\theta^2 D^2 t^{2\theta}.$$

Hence

$$F = \sup_{t>0} t^{-\theta} K_t(x) \leq N_\theta D.$$

If $D = \infty$, this inequality is void.

It remains to prove the reverse estimate. If $\lim_{r\to\infty} d(r) > 0$, then both D and F are infinite: the former is immediate from the definition of D, and the latter follows from (2) by letting $t \downarrow 0$. We may therefore assume that $d(r) \to 0$ as $r \to \infty$.

Let $r > 0$ be such that $d(r) > 0$. Since d is nonnegative, convex, and nonincreasing, the function $h = d^2$ is convex and nonincreasing. Choose a subgradient $p \in \partial h(r)$. Since h is nonincreasing, every subgradient is nonpositive. Moreover no subgradient can be zero: if $0 \in \partial h(r)$, convexity would imply $h(s) \geq h(r) > 0$ for all $s \geq 0$, contradicting $h(s) \to 0$ as $s \to \infty$. Hence $p < 0$. Define

$$t^2 := -\frac{p}{2r} > 0.$$

The supporting-line inequality for h gives, for all $s \geq 0$,

$$h(s) \geq h(r) + p(s - r).$$

Consequently,

$$h(s) + t^2 s^2 \geq h(r) + p(s-r) + t^2 s^2 = h(r) + t^2 r^2 + t^2 (s-r)^2 \geq h(r) + t^2 r^2.$$

Thus r minimizes $s \mapsto h(s) + t^2 s^2$, and (2) gives

$$K_t(x)^2 = d(r)^2 + t^2 r^2.$$

With $u := tr/d(r) > 0$, it follows that

$$t^{-\theta} K_t(x) = d(r)^{1-\theta} r^\theta u^{-\theta} (1 + u^2)^{1/2}.$$

By the definition of N_θ in equation (4),

$$\inf_{u>0} u^{-\theta}(1 + u^2)^{1/2} = N_\theta.$$

Therefore

$$F \geq t^{-\theta} K_t(x) \geq N_\theta d(r)^{1-\theta} r^\theta.$$

Taking the supremum over all $r > 0$ with $d(r) > 0$ yields

$$F \geq N_\theta D.$$

Together with the first inequality, this proves (1).

Erratum E4: the range of γ in Lemma 2.1

In Lemma 2.1, the quantifier in inequality (13) should be $\gamma > \nu$, not merely $\gamma > 0$. Thus the corrected statement is

$$\mu([0,\Lambda)) + \Lambda^{2\gamma} \int_{[\Lambda,\infty)} \lambda^{-2\gamma}\, d\mu(\lambda) \le \frac{\gamma}{\gamma-\nu} \Lambda^{2\nu} |||\mu|||_\nu^2 \qquad \forall \Lambda > 0, \qquad \forall \gamma > \nu. \tag{3}$$

The printed range cannot be correct. If $0 < \gamma < \nu$, then the factor $\gamma/(\gamma-\nu)$ is negative, while the left-hand side of inequality (13) is nonnegative. If $\gamma = \nu$, the displayed constant is undefined, and no uniform estimate of this form follows from $|||\mu|||_\nu < \infty$ alone. For example, on $[0,\infty)$ consider

$$d\mu(\lambda) = 2\nu\lambda^{2\nu-1} \mathbf{1}_{(0,1)}(\lambda)\, d\lambda.$$

Then $|||\mu|||_\nu = 1$, but for $0 < \Lambda < 1$,

$$\Lambda^{2\nu} \int_{[\Lambda,\infty)} \lambda^{-2\nu}\, d\mu(\lambda) = 2\nu\Lambda^{2\nu} \log(1/\Lambda),$$

so the ratio of the left-hand side to $\Lambda^{2\nu}$ is unbounded as $\Lambda \downarrow 0$.

We recall the proof of the corrected assertion. Let

$$I(t) := \mu([0,t)), \qquad M := |||\mu|||_\nu^2.$$

Then $I(t) \le t^{2\nu} M$ for all $t > 0$. For $\gamma > \nu$, integration by parts gives

$$\int_{[\Lambda,\infty)} \lambda^{-2\gamma}\, d\mu(\lambda) = -\Lambda^{-2\gamma} I(\Lambda) + 2\gamma \int_\Lambda^\infty \lambda^{-2\gamma-1} I(\lambda)\, d\lambda.$$

Using $I(\lambda) \le \lambda^{2\nu} M$ under the integral yields

$$\begin{aligned} \int_{[\Lambda,\infty)} \lambda^{-2\gamma}\, d\mu(\lambda) &\le -\Lambda^{-2\gamma} I(\Lambda) + 2\gamma M \int_\Lambda^\infty \lambda^{2\nu-2\gamma-1}\, d\lambda \\ &= -\Lambda^{-2\gamma} I(\Lambda) + \frac{\gamma}{\gamma-\nu} \Lambda^{2\nu-2\gamma} M. \end{aligned}$$

Multiplying by $\Lambda^{2\gamma}$ and adding $I(\Lambda)$ gives (3). The convergence of the integral

$$\int_\Lambda^\infty \lambda^{2\nu-2\gamma-1}\, d\lambda$$

is exactly the point at which the condition $\gamma > \nu$ is needed.

This correction does not affect the later use of inequality (13) in Proposition 2.2, where the hypotheses already impose $0 \le \nu < \gamma$.

Erratum E5: the cutoff power in Proposition 2.2

In the proof of Proposition 2.2, specifically in the proof of inequality (26a), the displayed estimate with the cutoff parameter s has the wrong homogeneity. The term

$$s^2 \int_{[s,\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda)$$

should be replaced, after the substitution $s = t^{1/\gamma}$, by

$$s^{2\gamma} \int_{[s,\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda).$$

Equivalently, before making the substitution, the factor should remain t^2.

We spell out the corrected argument. From equation (24), for every $t > 0$ and every cutoff $\rho > 0$,

$$|K_t^\gamma(x)|^2 = \int_{[0,\infty)} \frac{t^2}{t^2+\lambda^{2\gamma}}\, d\mu_x(\lambda)$$
$$\leq \mu_x([0,\rho)) + t^2 \int_{[\rho,\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda).$$

Indeed, on $[0,\rho)$ the integrand is at most 1, while on $[\rho,\infty)$ it is at most $t^2\lambda^{-2\gamma}$. Hence, choosing $\rho = t^{1/\gamma}$,

$$\|x\|_{\nu:\gamma}^2 = \sup_{t>0} t^{-2\nu/\gamma} |K_t^\gamma(x)|^2$$
$$\leq \sup_{t>0} t^{-2\nu/\gamma} \Big\{ \mu_x([0,t^{1/\gamma})) + t^2 \int_{[t^{1/\gamma},\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda) \Big\}.$$

With $s := t^{1/\gamma}$, this becomes

$$\|x\|_{\nu:\gamma}^2 \leq \sup_{s>0} s^{-2\nu} \Big\{ \mu_x([0,s)) + s^{2\gamma} \int_{[s,\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda) \Big\}.$$

Using inequality (13), with the corrected restriction $\gamma > \nu$ from Erratum E4, gives

$$\mu_x([0,s)) + s^{2\gamma} \int_{[s,\infty)} \lambda^{-2\gamma}\, d\mu_x(\lambda) \leq \frac{\gamma}{\gamma-\nu} s^{2\nu} |||x|||_\nu^2.$$

Therefore

$$\|x\|_{\nu:\gamma}^2 \leq \frac{\gamma}{\gamma-\nu} |||x|||_\nu^2,$$

which is exactly inequality (26a), i.e.

$$\sqrt{1-\nu/\gamma}\, \|x\|_{\nu:\gamma} \leq |||x|||_\nu.$$

Thus the statement of Proposition 2.2 is unchanged; only the homogeneous cutoff term in its proof must be corrected.

Erratum E6: the endpoint $\nu = 0$ in Proposition 2.2

In the proof of Proposition 2.2, specifically in the proof of inequality (26b), the displayed choice

$$t = s^\gamma \sqrt{\frac{\gamma-\nu}{\nu}}$$

is meaningful only for $0 < \nu < \gamma$. The endpoint $\nu = 0$ should be handled separately.

For $0 < \nu < \gamma$, the printed argument is valid. Indeed, putting $\theta := \nu/\gamma$, the displayed choice of t is exactly the value for which

$$s^{-2\nu} = N_\theta^2 t^{-2\theta} \frac{t^2}{t^2+s^{2\gamma}}.$$

For $\lambda \leq s$ this gives

$$s^{-2\nu} \leq N_\theta^2 t^{-2\theta} \frac{t^2}{t^2+\lambda^{2\gamma}},$$

and hence, using equation (24),

$$s^{-2\nu} \|E_{[0,s)}x\|^2 \leq N_\theta^2 t^{-2\theta} |K_t^\gamma(x)|^2 \leq N_\theta^2 \|x\|_{\nu:\gamma}^2.$$

Taking the supremum over $s > 0$ yields

$$|||x|||_\nu \leq N_{\nu/\gamma} \|x\|_{\nu:\gamma}$$

for $0 < \nu < \gamma$.

It remains only to record the endpoint $\nu = 0$. Since $N_0 = 1$, the claim becomes

$$|||x|||_0 \le \|x\|_{0:\gamma}.$$

In fact equality holds. First,

$$|||x|||_0 = \sup_{s>0} \|E_{[0,s)}x\| = \|x\|.$$

The last equality follows because the spectral measure of T^*T is compactly supported and $E_{[0,s)}$ equals the identity for all sufficiently large s. On the other hand, by equation (24),

$$|K_t^\gamma(x)|^2 = \int_{[0,\infty)} \frac{t^2}{t^2+\lambda^{2\gamma}}\, d\mu_x(\lambda) \le \int_{[0,\infty)} d\mu_x(\lambda) = \|x\|^2.$$

Thus $\|x\|_{0:\gamma} \le \|x\|$. Conversely, as $t \to \infty$ the integrand in equation (24) increases pointwise to 1; by monotone convergence,

$$\lim_{t\to\infty} |K_t^\gamma(x)|^2 = \|x\|^2.$$

Therefore

$$\|x\|_{0:\gamma} = \sup_{t>0} |K_t^\gamma(x)| = \|x\| = |||x|||_0.$$

This proves inequality (26b) also at $\nu = 0$. Thus the statement of Proposition 2.2 is unchanged; only the endpoint argument needs to be supplied separately.

Erratum E7: the restriction $\gamma > 0$ when ν/γ appears

Several definitions and statements use the quotient ν/γ. They should therefore be read with the restriction $\gamma > 0$, unless a separate convention is explicitly introduced for the degenerate case $\gamma = 0$.

In particular, after the definition of X_γ in equation (21), the interpolation spaces should be introduced as follows:

$$\text{for } \gamma > 0 \text{ and } 0 \le \nu \le \gamma, \qquad X_{\nu:\gamma} := (X, X_\gamma)_{\nu/\gamma,\infty}.$$

The case $\gamma = 0$ is not covered by this formula, since then necessarily $\nu = 0$ but the expression ν/γ is the indeterminate quotient $0/0$. If desired, one may add the separate convention

$$X_{0:0} := X, \qquad \|\cdot\|_{0:0} := \|\cdot\|.$$

This convention is not needed for the subsequent arguments.

The same restriction is needed in Proposition 2.4. Its statement should begin

$$\text{Let } \gamma > 0, \quad 0 \le \nu \le \gamma, \quad \text{and } x \in X.$$

Indeed, the displayed identity in Proposition 2.4 contains the three quantities

$$\|(T^*T)^\gamma \omega\|^{-\nu/\gamma}, \qquad N_{\nu/\gamma}, \qquad \|x\|_{\nu:\gamma},$$

all of which are undefined as written when $\gamma = 0$.

With the correction $\gamma > 0$, the proposition itself is unchanged. The endpoint cases discussed in its proof are then the meaningful endpoints $\nu = 0$ and $\nu = \gamma$ for a fixed positive γ. If one separately adopts the convention $X_{0:0} = X$, then the missing degenerate case $\gamma = \nu = 0$ reduces only to the elementary identity

$$\sup_{\|\omega\|=1} |\langle x, \omega\rangle| = \|x\|.$$

Thus this erratum is a domain-of-notation correction: it does not change any of the nondegenerate estimates or their proofs.

Erratum E8: the lower-bound proof in Proposition 2.4

The lower-bound part of the proof of Proposition 2.4 invokes the identity (9). Since the E-part of (9) is false, that part of the proof should either use only the corrected distance-function identity from Erratum E3, or be replaced by a direct Hilbert-space duality argument. We record such a replacement here.

The first half of the proof in the paper already proves the inequality "$\le$" in equation (36). It remains only to justify the reverse inequality in the interior case $0 < \nu < \gamma$. Put

$$\theta := \frac{\nu}{\gamma}, \qquad B := (T^*T)^{\gamma}.$$

For nonzero $\omega \in X$ define

$$F(x) := \sup_{\omega \neq 0} \frac{|\langle x, \omega \rangle|}{\|\omega\|^{1-\theta} \|B\omega\|^{\theta}}.$$

By homogeneity this is the same quantity as the left-hand side of equation (36). If $F(x) = \infty$, there is nothing to prove, so assume $F(x) < \infty$.

We use the following dual representation of the K-functional. From equation (24), or equivalently from the spectral theorem,

$$|K_t^{\gamma}(x)|^2 = \langle M_t x, x \rangle, \qquad M_t := t^2(t^2 + B^2)^{-1}.$$

Since B is bounded and self-adjoint, M_t is bounded, positive, and boundedly invertible, with

$$M_t^{-1} = I + t^{-2}B^2.$$

Therefore

$$\begin{aligned} K_t^{\gamma}(x) &= \sup_{\omega \neq 0} \frac{|\langle x, \omega \rangle|}{\left(\langle M_t^{-1}\omega, \omega \rangle\right)^{1/2}} \\ &= \sup_{\omega \neq 0} \frac{|\langle x, \omega \rangle|}{\left(\|\omega\|^2 + t^{-2}\|B\omega\|^2\right)^{1/2}}. \end{aligned} \tag{4}$$

Indeed, the identity is the elementary Hilbert-space formula

$$\langle M_t x, x \rangle^{1/2} = \sup_{\omega \neq 0} \frac{|\langle x, \omega \rangle|}{\langle M_t^{-1}\omega, \omega \rangle^{1/2}}.$$

By the definition of $F(x)$,

$$|\langle x, \omega \rangle| \le F(x) \|\omega\|^{1-\theta} \|B\omega\|^{\theta}.$$

Using Young's inequality in the form of equation (5), with $a = \|\omega\|^2$ and $b = t^{-2}\|B\omega\|^2$, gives

$$N_{\theta} \|\omega\|^{1-\theta} \left(t^{-1}\|B\omega\|\right)^{\theta} \le \left(\|\omega\|^2 + t^{-2}\|B\omega\|^2\right)^{1/2}.$$

Equivalently,

$$\|\omega\|^{1-\theta} \|B\omega\|^{\theta} \le N_{\theta}^{-1} t^{\theta} \left(\|\omega\|^2 + t^{-2}\|B\omega\|^2\right)^{1/2}.$$

Substituting this into (4) yields, for every $t > 0$,

$$K_t^{\gamma}(x) \le N_{\theta}^{-1} F(x) t^{\theta}.$$

Consequently,

$$\|x\|_{\nu:\gamma} = \sup_{t>0} t^{-\theta} K_t^{\gamma}(x) \le N_{\theta}^{-1} F(x).$$

Thus

$$N_{\nu/\gamma} \|x\|_{\nu:\gamma} \le F(x),$$

which is precisely the missing inequality "$\ge$" in equation (36). Combined with the already proved opposite inequality, this gives

$$F(x) = N_{\nu/\gamma} \|x\|_{\nu:\gamma}.$$

This argument replaces the appeal to the variational-source-condition results and avoids any use of the false E-identity in (9). The endpoint cases $\nu = 0$ and $\nu = \gamma$ remain as stated in the original proof, subject only to the restriction $\gamma > 0$ noted in Erratum E7.

Erratum E9: the auxiliary Tikhonov problem in equations (55)–(56)

Equations (55)–(56) contain two related misprints. First, the second term in equation (56),

$$\alpha\|x^\dagger - x_0\|^2,$$

is independent of the minimization variable x. Thus the printed minimization problem is not a Tikhonov functional. For instance, when $\gamma > 0$, the injectivity of T implies the injectivity of $(T^*T)^{\gamma/2}$, and the printed minimizer is simply $x_{\alpha:\gamma} = x^\dagger$. Hence the middle term in equation (55) would be zero, which is impossible in general.

Second, even after replacing $\|x^\dagger - x_0\|^2$ by $\|x - x_0\|^2$, the power of the parameter in equation (55) has to be adjusted. The corrected statement can be written in either of the following two equivalent forms.

With a parameter $\beta > 0$, define

$$x_{\beta:\gamma} := \operatorname*{argmin}_{x\in X} \left\{ \|(T^*T)^{\gamma/2}(x^\dagger - x)\|^2 + \beta\|x - x_0\|^2 \right\}. \tag{5}$$

Then, for $\gamma > 0$ and $0 \le \nu \le \gamma$,

$$N_{\nu/\gamma}^{-1}\|x^\dagger - x_0\|_{\nu:\gamma} \le \sup_{\beta>0} \beta^{-\nu/\gamma}\|x^\dagger - x_{\beta:\gamma}\| \le \|x^\dagger - x_0\|_{\nu:\gamma}. \tag{6}$$

Equivalently, if one wants to keep the factor $\alpha^{-\nu}$ appearing in equation (55), then equation (56) should be replaced by

$$x_{\alpha:\gamma} := \operatorname*{argmin}_{x\in X} \left\{ \|(T^*T)^{\gamma/2}(x^\dagger - x)\|^2 + \alpha^\gamma\|x - x_0\|^2 \right\}. \tag{7}$$

With this definition, the estimate in equation (55) is valid as printed.

We include the verification. Put

$$A := T^*T, \qquad B := A^{\gamma/2}, \qquad e := x^\dagger - x_0.$$

For the corrected problem (5), the normal equation is

$$(B^*B + \beta I)x_{\beta:\gamma} = B^*Bx^\dagger + \beta x_0.$$

Since $B^*B = A^\gamma$, this is

$$(A^\gamma + \beta I)x_{\beta:\gamma} = A^\gamma x^\dagger + \beta x_0.$$

Subtracting this equation from

$$(A^\gamma + \beta I)x^\dagger = A^\gamma x^\dagger + \beta x^\dagger$$

gives

$$(A^\gamma + \beta I)(x^\dagger - x_{\beta:\gamma}) = \beta(x^\dagger - x_0),$$

and therefore

$$x^\dagger - x_{\beta:\gamma} = \beta(A^\gamma + \beta I)^{-1}e. \tag{8}$$

This is exactly the ordinary Tikhonov error formula for the operator $B = (T^*T)^{\gamma/2}$.

For this operator B, the endpoint space in Proposition 3.1 is

$$\operatorname{range}(B^*B) = \operatorname{range}(A^\gamma) = X_\gamma,$$

with the same norm, since

$$\|z\|_{1,B} = \|(B^*B)^{-1}z\| = \|A^{-\gamma}z\| = \|z\|_\gamma.$$

Thus Proposition 3.1, applied to B with interpolation exponent ν/γ, gives precisely (6). Finally, setting

$$\beta = \alpha^\gamma$$

turns $\beta^{-\nu/\gamma}$ into $\alpha^{-\nu}$ and converts (5) into (7). This proves equation (55) with the corrected definition of $x_{\alpha:\gamma}$.

Erratum E10: endpoint cases in Proposition 3.2

The proof of Proposition 3.2 says that the first inequality in inequality (59) is a consequence of equation (6) with $\theta = \nu/k$. This is correct in the interior case $0 < \nu < k$. At the endpoints $\nu = 0$ and $\nu = k$, however, the displayed choice of the parameter in equation (6) degenerates, as noted in Erratum E1. The statement of Proposition 3.2 remains correct, but the endpoint cases should be justified separately.

Put

$$e := x^\dagger - x_0, \qquad A := T^*T, \qquad S_\alpha := \alpha(A+\alpha I)^{-1}.$$

By equation (44),

$$x^\dagger - x_{\alpha,k} = S_\alpha^k e.$$

The spectral representation used in the proof is

$$\alpha^{-2\nu}\|S_\alpha^k e\|^2 = \alpha^{-2\nu}\int_{[0,\infty)} \Big(\frac{\alpha}{\alpha+\lambda}\Big)^{2k} d\mu_e(\lambda). \tag{9}$$

First let $\nu = 0$. Then $N_0 = 1$ and, by the identity $X_{0:k} = X$ from equation (25),

$$\|e\|_{0:k} = \|e\|.$$

For every $\alpha > 0$ the multiplier $\alpha/(\alpha+\lambda)$ is bounded by 1, so

$$\|S_\alpha^k e\| \le \|e\|.$$

On the other hand,

$$\Big(\frac{\alpha}{\alpha+\lambda}\Big)^{2k} \longrightarrow 1 \qquad (\alpha \to \infty)$$

for every $\lambda \ge 0$, and the integrand is bounded by 1. Hence Lebesgue dominated convergence gives

$$\lim_{\alpha\to\infty} \|S_\alpha^k e\|^2 = \int_{[0,\infty)} d\mu_e(\lambda) = \|e\|^2.$$

Thus

$$\sup_{\alpha>0} \|x^\dagger - x_{\alpha,k}\| = \|e\| = \|e\|_{0:k},$$

which proves inequality (59) at $\nu = 0$, in fact with equality.

Now let $\nu = k$. Then $N_1 = 1$ and, again by the identity $X_{k:k} = X_k$ from equation (25),

$$\|e\|_{k:k} = \|e\|_k.$$

From (9),

$$\alpha^{-2k}\|S_\alpha^k e\|^2 = \int_{[0,\infty)} (\alpha+\lambda)^{-2k} d\mu_e(\lambda). \tag{10}$$

Since T is injective, $E_{\{0\}} = 0$, hence $\mu_e(\{0\}) = 0$. For $\lambda > 0$ the function $(\alpha+\lambda)^{-2k}$ increases to λ^{-2k} as $\alpha \searrow 0$. Therefore the monotone convergence theorem, with the value $+\infty$ allowed, gives

$$\lim_{\alpha\searrow 0} \alpha^{-2k}\|S_\alpha^k e\|^2 = \int_{[0,\infty)} \lambda^{-2k}\, d\mu_e(\lambda) = \|e\|_k^2.$$

Consequently,

$$\sup_{\alpha>0} \alpha^{-k}\|x^\dagger - x_{\alpha,k}\| = \lim_{\alpha\searrow 0} \alpha^{-k}\|x^\dagger - x_{\alpha,k}\| = \|e\|_k = \|e\|_{k:k}.$$

This proves inequality (59) at $\nu = k$ and also justifies the final sentence of Proposition 3.2.

Erratum E11: the residual representation in equation (64)

The residual representation in equation (64) is missing the factor σ. The corrected formula is

$$y^\dagger - Tx_k = (I - \sigma TT^*)^k (y^\dagger - Tx_0), \tag{11}$$

$$y^\delta - Tx_k^\delta = (I - \sigma TT^*)^k (y^\delta - Tx_0). \tag{12}$$

The formula printed in equation (64) is the special case $\sigma = 1$.

Indeed, put

$$r_k^\dagger := y^\dagger - Tx_k.$$

Using the Landweber iteration

$$x_k = x_{k-1} + \sigma T^*(y^\dagger - Tx_{k-1}),$$

$$\begin{aligned} r_k^\dagger &= y^\dagger - Tx_{k-1} - \sigma TT^*(y^\dagger - Tx_{k-1}) \\ &= (I - \sigma TT^*) r_{k-1}^\dagger. \end{aligned}$$

Thus induction gives the first identity in (11). Replacing $y^\dagger$ by y^δ gives the second identity.

The later residual comparison used in the discrepancy-principle paragraph remains valid after this correction. Subtracting the two identities in (11) gives

$$(y^\dagger - Tx_k) - (y^\delta - Tx_k^\delta) = (I - \sigma TT^*)^k (y^\dagger - y^\delta).$$

Since TT^* is nonnegative self-adjoint and

$$\sigma \|TT^*\| = \sigma \|T^*T\| \le 1$$

by the stepsize condition

$$0 < \sigma \|T^*T\| \le 1,$$

we have

$$\|I - \sigma TT^*\| \le 1.$$

Consequently, whenever the noise condition

$$\|y^\dagger - y^\delta\| \le \delta$$

holds,

$$\begin{aligned} \left| \|y^\dagger - Tx_k\| - \|y^\delta - Tx_k^\delta\| \right| &\le \|(y^\dagger - Tx_k) - (y^\delta - Tx_k^\delta)\| \\ &\le \|y^\dagger - y^\delta\| \le \delta. \end{aligned}$$

Thus the estimate following equation (64) is correct, but it should be derived from the corrected residual formula (11).

Erratum E12: the noisy Landweber splitting in equation (65)

For a general Landweber step size σ, the noise-propagation term in equation (65) should contain a factor $\sqrt{\sigma}$. The corrected estimate is

$$\|x^\dagger - x_k^\delta\| \le \|x^\dagger - x_k\| + \delta \sqrt{\sigma k} \qquad \forall k \ge 0. \tag{13}$$

The formula printed in equation (65) is the special case $\sigma = 1$; it also follows from (13) if one has the additional normalization $\sigma \le 1$.

Let

$$z_k := x_k - x_k^\delta, \qquad q := y^\dagger - y^\delta, \qquad A := T^*T.$$

Then $z_0 = 0$, and subtracting the noisy and the exact Landweber iterations

$$x_k = x_{k-1} + \sigma T^*(y^\dagger - Tx_{k-1})$$

and its noisy analogue gives

$$z_k = (I - \sigma A) z_{k-1} + \sigma T^* q.$$

Hence

$$z_k = \sigma \sum_{j=0}^{k-1} (I - \sigma A)^j T^* q. \tag{14}$$

By the functional calculus, the operator in (14) has norm

$$\sup_{0 \le \lambda \le \|A\|} \sqrt{\lambda}\, \sigma \sum_{j=0}^{k-1} (1 - \sigma\lambda)^j.$$

Using the stepsize condition

$$0 < \sigma \|T^* T\| \le 1,$$

put $s := \sigma\lambda \in [0, 1]$. Then, for $\lambda > 0$,

$$\sqrt{\lambda}\, \sigma \sum_{j=0}^{k-1} (1 - \sigma\lambda)^j = \sqrt{\sigma}\, \frac{1 - (1-s)^k}{\sqrt{s}}.$$

For $0 \le s \le 1$,

$$\begin{aligned} \left(1 - (1-s)^k\right)^2 &= \left(s \sum_{j=0}^{k-1} (1-s)^j \right)^2 \\ &\le s^2 k \sum_{j=0}^{k-1} (1-s)^{2j} \le \frac{ks}{2-s} \le ks. \end{aligned}$$

Therefore

$$\left\| \sigma \sum_{j=0}^{k-1} (I - \sigma A)^j T^* \right\|_{Y \to X} \le \sqrt{\sigma k}.$$

Together with the noise condition

$$\|y^\dagger - y^\delta\| \le \delta,$$

this gives

$$\|x_k - x_k^\delta\| = \|z_k\| \le \sqrt{\sigma k}\, \|y^\dagger - y^\delta\| \le \delta \sqrt{\sigma k}.$$

The triangle inequality then yields (13).

The factor $\sqrt{\sigma}$ cannot be omitted under the sole stepsize assumption

$$0 < \sigma \|T^* T\| \le 1.$$

For example, take $X = Y = \mathbb{R}$, $T = \varepsilon I$, $0 < \varepsilon < 1$, $k = 1$, and $\sigma = \varepsilon^{-2}$. Then $\sigma \|T^* T\| = 1$, while the propagated noise in one step has size $\sigma \|T^* q\| = \delta / \varepsilon = \delta \sqrt{\sigma}$. This exceeds δ when $\varepsilon < 1$.

Erratum E13: the parameter range in equation (67)

The statement surrounding equation (67) should require $\nu > 0$, or else an explicit limiting convention should be imposed. As printed, the factor

$$(1 + k/\nu)^\nu$$

is undefined when $\nu = 0$. Thus the sentence following equation (67) should read that the hidden constants may depend on the fixed parameter $\nu > 0$.

This is only a parameter-range correction. It is consistent with Proposition 3.4, which is also stated for $\nu > 0$. Indeed, when $r = 0$ the normalization in Proposition 3.4 is

$$\varepsilon_k = \frac{1}{\sigma} \frac{\nu}{k + \nu},$$

and therefore

$$\varepsilon_k^{-\nu} = \sigma^{\nu}(1+k/\nu)^{\nu}.$$

Since σ is fixed, the factor in equation (67) is just the Landweber rate normalization from Proposition 3.4, up to the constant σ^{ν}. This normalization is meaningful for $\nu > 0$.

If one wants to include the endpoint $\nu = 0$ formally, the natural convention is the limit

$$\lim_{\nu \downarrow 0}(1+k/\nu)^{\nu} = 1 \qquad (k \text{ fixed}),$$

and then equation (67) becomes the boundedness comparison

$$\sup_{\|y^{\dagger}-y^{\delta}\|\le\delta} \|x^{\dagger} - x^{\delta}_{\bar{k}(\delta,y^{\delta},\dots)}\| \sim \sup_{k\ge 0} \|x^{\dagger} - x_k\|.$$

This endpoint convention, however, is separate from the algebraic-rate case treated in Proposition 3.4. The simplest correction is therefore to state equation (67) only for $\nu > 0$.

Erratum E14: the maximization domain in the proof of Proposition 3.4

In the proof of Proposition 3.4, the line preceding the normalization $x_0 = 0$ and $\sigma = 1$ should not maximize over all $\lambda \ge 0$. The correct statement is that

$$\varepsilon_k = \operatorname*{argmax}_{0\le\lambda\le 1/\sigma} \lambda^{2(r+\nu)}(1-\sigma\lambda)^{2k}, \qquad k \ge 0. \tag{15}$$

Equivalently, after the normalization $\sigma = 1$, the maximization is over $0 \le \lambda \le 1$. This is the relevant interval, since the step-size condition

$$0 < \sigma\|T^*T\| \le 1$$

implies

$$\operatorname{supp} \mu_{x^{\dagger}-x_0} \subset [0, \|T^*T\|] \subset [0, 1/\sigma].$$

The formulation with $\lambda \ge 0$ is false: outside the stability interval $[0, 1/\sigma]$ the factor $(1-\sigma\lambda)^{2k}$ grows again, and the displayed function is not maximized at ε_k.

We verify (15). Put

$$a := r + \nu > 0, \qquad f_k(\lambda) := \lambda^{2a}(1-\sigma\lambda)^{2k}.$$

For $k = 0$, $f_0(\lambda) = \lambda^{2a}$ is maximized on $[0, 1/\sigma]$ at $\lambda = 1/\sigma$, which is precisely

$$\varepsilon_0 = \frac{1}{\sigma}\frac{a}{a} = \frac{1}{\sigma}.$$

For $k \ge 1$, consider $0 < \lambda < 1/\sigma$. Then

$$\frac{d}{d\lambda}\log f_k(\lambda) = \frac{2a}{\lambda} - \frac{2k\sigma}{1-\sigma\lambda}.$$

The unique critical point is therefore determined by

$$a(1-\sigma\lambda) = k\sigma\lambda,$$

that is,

$$\lambda = \frac{1}{\sigma}\frac{a}{k+a} = \frac{1}{\sigma}\frac{r+\nu}{k+r+\nu} = \varepsilon_k.$$

Moreover,

$$\frac{d^2}{d\lambda^2}\log f_k(\lambda) = -\frac{2a}{\lambda^2} - \frac{2k\sigma^2}{(1-\sigma\lambda)^2} < 0,$$

so this critical point is the unique maximum in the open interval. Since $f_k(0) = f_k(1/\sigma) = 0$ for $k \ge 1$, it is also the maximum on $[0, 1/\sigma]$.

This correction does not affect the proof of Proposition 3.4. All spectral integrals used there are supported in $[0, \|T^*T\|] \subset [0, 1/\sigma]$, and the subsequent estimates use only this stability interval.

Erratum E15: the sufficient inequality in the discrepancy-principle paragraph

In the paragraph following Proposition 3.4, the displayed sufficient condition for the discrepancy principle has the inequality in the wrong direction. The corrected condition is

$$c_2\,\varepsilon_k^{\nu+1/2}|||x^\dagger - x_0|||_\nu \le \delta(\tau-1). \tag{16}$$

The printed condition has the two sides of (16) reversed.

Indeed, using the upper estimate in Proposition 3.4 with $r = 1/2$ gives

$$\|y^\dagger - Tx_k\| = \|T(x^\dagger - x_k)\| = \|(T^*T)^{1/2}(x^\dagger - x_k)\| \tag{17}$$

$$\le c_2\,\varepsilon_k^{\nu+1/2}|||x^\dagger - x_0|||_\nu. \tag{18}$$

Therefore (16) implies

$$\|y^\dagger - Tx_k\| \le \delta(\tau-1).$$

Together with the residual comparison established just before this paragraph,

$$\big|\|y^\dagger - Tx_k\| - \|y^\delta - Tx_k^\delta\|\big| \le \delta,$$

this yields

$$\|y^\delta - Tx_k^\delta\| \le \|y^\dagger - Tx_k\| + \delta \le \delta\tau,$$

which is precisely the discrepancy principle in equation (68). The printed reverse inequality does not imply the desired residual estimate; it only says that the available upper bound in (17) is at least the target threshold.

The corrected inequality also gives the same iteration-count estimate as claimed in the text. For $r = 1/2$,

$$\varepsilon_k = \frac{1}{\sigma}\frac{\nu+1/2}{k+\nu+1/2}.$$

Put

$$a := \nu + \frac{1}{2}, \qquad R := |||x^\dagger - x_0|||_\nu.$$

If $R = 0$ the assertion is trivial. Otherwise, (16) is equivalent to

$$c_2 R\left(\frac{a}{\sigma(k+a)}\right)^a \le \delta(\tau-1),$$

or

$$k + a \ge \frac{a}{\sigma}\left(\frac{c_2 R}{\delta(\tau-1)}\right)^{1/a}.$$

Thus the discrepancy principle is met after at most

$$k^\star = O\big(\delta^{-1/a}\big) = O\big(\delta^{-2/(2\nu+1)}\big)$$

iterations, with constants depending on the fixed quantities σ, τ, ν, c_2, and $|||x^\dagger - x_0|||_\nu$. Thus only the direction of the displayed sufficient inequality has to be corrected.

Erratum E16: equation (84) in the general-σ normalization

Equation (84), immediately following Proposition 3.4, is stated in the normalization $\sigma = 1$. For a general Landweber step size σ, the corresponding form is obtained by restoring the factor $\sigma^{-(r+\nu)}$.

Indeed, the upper estimate in Proposition 3.4 gives, for each fixed $k \ge 0$,

$$\|(T^*T)^r(x^\dagger - x_k)\| \le c_2\,\varepsilon_k^{r+\nu}|||x^\dagger - x_0|||_\nu, \tag{19}$$

where

$$\varepsilon_k = \frac{1}{\sigma}\frac{r+\nu}{k+r+\nu}.$$

Thus

$$\varepsilon_k^{r+\nu} = \sigma^{-(r+\nu)} \left(\frac{r+\nu}{k+r+\nu} \right)^{r+\nu}. \tag{20}$$

Putting $a := r + \nu$, and keeping r and k fixed as $\nu \to \infty$, one has

$$\begin{aligned} \left(\frac{a}{a+k} \right)^a &= \left(1 + \frac{k}{a} \right)^{-a} \\ &= \exp\left(-a \log\left(1 + \frac{k}{a} \right) \right) \\ &= \exp\left(-k + \frac{k^2}{2a} + \mathcal{O}(a^{-2}) \right) \\ &= e^{-k} \left(1 + \frac{k^2}{2\nu} + \mathcal{O}(\nu^{-2}) \right). \end{aligned}$$

Consequently, for general σ, equation (84) should read

$$\|(T^*T)^r (x^\dagger - x_k)\| \le c_2 \, \sigma^{-(r+\nu)} e^{-k} \left(1 + \frac{1}{2} k^2 \nu^{-1} + \mathcal{O}(\nu^{-2}) \right) |||x^\dagger - x_0|||_\nu \quad \text{as } \nu \to \infty, \tag{21}$$

for fixed r and fixed k. When $\sigma = 1$, (21) reduces to the display printed in the paper.