Theorem 5.2.18 errata

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Abstract

This note records errata to Theorem 5.2.18 in

> RA, *Stability of space-time Petrov-Galerkin discretizations for parabolic evolution equations*, 2012, doi:10.3929/ethz-a-007563932.

The theorem's conclusion remains the uniform discrete inf-sup estimate

$$\inf_{L\in\mathbb{N}_0} \gamma_B(X_L, Y_L) > 0.$$

The corrections below make explicit the intended meaning of the sparse tensor notation, add the closedness convention needed in the abstract statement, and fill in the omitted transfer from the auxiliary test space to the actual test space.

1. Corrections to the surrounding notation

C1. Sparse tensor symbols. The symbols $\bigcup$ in Proposition 4.4.12, Proposition 5.2.14, Remark 5.2.15, and Theorem 5.2.18 should be read as linear sums, not literal set-theoretic unions. Thus, for example,

$$X_L = \sum_{0\le k+\ell\le L} E_k \otimes U_\ell, \qquad Y_{L,1} = \sum_{0\le k+\ell\le L} F_k \otimes V_\ell.$$

A literal union of subspaces need not be a subspace.

C2. Auxiliary temporal spaces. In the proof of Theorem 5.2.18 the auxiliary temporal space $E_k + \partial_t E_k$ should be replaced, in the infinite-dimensional abstract formulation, by its $L^2(J)$-closure:

$$\widehat{F}_k := \overline{E_k + \partial_t E_k}^{L^2(J)}, \qquad \dot{E}_k := \overline{\partial_t E_k}^{L^2(J)}.$$

For the finite-dimensional spline and wavelet spaces used later in the thesis, the closures are unnecessary.

C3. Auxiliary test space. The auxiliary test space should be written as

$$\widehat{Y}_L = \widehat{Y}_{L,1} \times V_L, \qquad \widehat{Y}_{L,1} := \sum_{0\le k+\ell\le L} \widehat{F}_k \otimes V_\ell.$$

If the printed expression is interpreted as a sum of product subspaces, its second component is V_L by nestedness of the V_ℓ.

2. Corrected theorem, basic sparse case

Corrected theorem. Let $(E_k, F_k)_{k\in\mathbb{N}_0}$ be an admissible temporal pair in the sense of Definition 5.2.17. Let

$$U_\ell \subseteq V_\ell \subseteq V, \qquad \ell \in \mathbb{N}_0,$$

be nested non-trivial closed spatial subspaces satisfying

$$\eta := \inf_{\ell\in\mathbb{N}_0} K_{V'\times V}(U_\ell, V_\ell) > 0.$$

Assume, in addition, either that all building blocks are finite-dimensional, or that the auxiliary spaces defined in C2–C3 and the finite sparse tensor sums below are closed subspaces of the ambient Hilbert spaces.

For $L \in \mathbb{N}_0$, define

$$X_L := \sum_{0\le k+\ell\le L} E_k \otimes U_\ell, \qquad Y_{L,1} := \sum_{0\le k+\ell\le L} F_k \otimes V_\ell, \qquad Y_L := Y_{L,1} \times V_L.$$

Then

$$\inf_{L\in\mathbb{N}_0} \gamma_B(X_L, Y_L) > 0.$$

The full tensor, anisotropic sparse tensor, and finite-sum variants in Remark 5.2.15 remain valid after the same changes.

3. Proof of the corrected theorem

Step 1: admissibility passes to the closed auxiliary space. Definition 5.2.17 gives

$$c_T^0 := \inf_{k\in\mathbb{N}_0} K_{L^2(J)\times L^2(J)}(E_k + \partial_t E_k, F_k) > 0.$$

Let $P_k : L^2(J) \to F_k$ be the $L^2(J)$-orthogonal projector. Since F_k is closed,

$$\|P_k f\|_{L^2(J)} \ge c_T^0 \|f\|_{L^2(J)} \qquad \forall f \in E_k + \partial_t E_k.$$

By continuity, the same estimate holds for all $f \in \widehat{F}_k$. Hence

$$c_T := \inf_{k\in\mathbb{N}_0} K_{L^2(J)\times L^2(J)}(\widehat{F}_k, F_k) \ge c_T^0 > 0.$$

Step 2: stability with the auxiliary test space. The corrected auxiliary space satisfies the inclusion hypothesis of Corollary 5.2.13:

$$X_L \times \{w_L(0) : w_L \in X_L\} \subseteq \widehat{Y}_L,$$

because $E_k \subseteq \widehat{F}_k$ and $U_\ell \subseteq V_\ell \subseteq V_L$ for $\ell \le L$.

If $\partial_t X_L \neq \{0\}$, apply Proposition 4.4.12 (equivalently, the tensor-product step behind Proposition 5.2.14) to the temporal pair $(\dot{E}_k, \widehat{F}_k)$, omitting any initial zero derivative levels. For every nonzero $\dot{E}_k$,

$$K_{L^2(J)\times L^2(J)}(\dot{E}_k, \widehat{F}_k) = 1$$

because $\dot{E}_k \subseteq \widehat{F}_k$. Therefore the tensor-product estimate gives

$$K_{Y_1'\times Y_1}\left(\sum_{0\le k+\ell\le L} \dot{E}_k \otimes U_\ell, \widehat{Y}_{L,1}\right) \ge \eta.$$

Since

$$\partial_t X_L \subseteq \sum_{0\le k+\ell\le L} \dot{E}_k \otimes U_\ell,$$

we also have

$$K_{Y_1'\times Y_1}(\partial_t X_L, \widehat{Y}_{L,1}) \ge \eta.$$

Corollary 5.2.13 then yields a constant $c_1 > 0$, independent of L, such that

$$\gamma_B(X_L, \widehat{Y}_L) \ge c_1$$

for all levels with $\partial_t X_L \neq \{0\}$.

If $\partial_t X_L = \{0\}$, then each $w \in X_L$ is time-independent. Testing with $v = (w, w(0)) \in \widehat{Y}_L$ and using $a_{\text{shift}} = 0$ gives

$$\langle Bw, v\rangle = \int_J a(t; w, w)\, dt + \|w(0)\|_H^2 \ge a_{\min}\|w\|^2_{L^2(J;V)}.$$

For such w, $\|w\|_X = \|w\|_{L^2(J;V)}$, while $\|v\|_Y \le C\|w\|_X$ with C depending only on J and the embedding $V \hookrightarrow H$. These degenerate levels therefore also satisfy a uniform lower bound. Consequently,

$$\inf_{L\in\mathbb{N}_0} \gamma_B(X_L, \widehat{Y}_L) > 0.$$

Step 3: transfer from $\widehat{Y}_L$ to Y_L. It remains to prove the step omitted in the printed proof:

$$\inf_{L\in\mathbb{N}_0} K_{Y\times Y}(\widehat{Y}_L, Y_L) > 0.$$

Set

$$W_0 := V_0, \qquad W_j := V_j \cap V_{j-1}^{\perp_V} \quad (j \ge 1).$$

Then $V_\ell = \bigoplus_{j=0}^{\ell} W_j$ orthogonally in V. By nestedness of the temporal spaces,

$$\widehat{Y}_{L,1} = \bigoplus_{j=0}^{L} \widehat{F}_{L-j} \otimes W_j, \qquad Y_{L,1} = \bigoplus_{j=0}^{L} F_{L-j} \otimes W_j.$$

Let $\Pi_L : Y_1 \to Y_{L,1}$ be the $Y_1 = L^2(J;V)$-orthogonal projector. For

$$z = \sum_{j=0}^{L} z_j, \qquad z_j \in \widehat{F}_{L-j} \otimes W_j,$$

the projector acts blockwise as

$$\Pi_L z = \sum_{j=0}^{L} (P_{L-j} \otimes I_{W_j}) z_j.$$

Using the lower bound from Step 1 and orthogonality of the W_j-blocks,

$$\begin{aligned}\|\Pi_L z\|_{Y_1}^2 &= \sum_{j=0}^{L} \|(P_{L-j} \otimes I_{W_j}) z_j\|_{Y_1}^2 \\ &\ge c_T^2 \sum_{j=0}^{L} \|z_j\|_{Y_1}^2 = c_T^2 \|z\|_{Y_1}^2.\end{aligned}$$

The projection characterization of K gives

$$K_{Y_1\times Y_1}(\widehat{Y}_{L,1}, Y_{L,1}) \geq c_T.$$

The second component is identical in $\widehat{Y}_L$ and Y_L, namely V_L. Hence

$$K_{Y\times Y}(\widehat{Y}_L, Y_L) \geq c_T.$$

Finally, the robustness estimate (4.4.34) gives

$$\gamma_B(X_L, Y_L) \geq K_{Y\times Y}(\widehat{Y}_L, Y_L)\,\gamma_B(X_L, \widehat{Y}_L) \geq c_T\,\gamma_B(X_L, \widehat{Y}_L).$$

Taking the infimum over L proves the corrected theorem.

4. Variants in Remark 5.2.15

For full tensor products, anisotropic sparse tensor products, and finite sums of sparse tensor products, the same proof applies after replacing the basic lower set

$$\{(k,\ell) : 0 \leq k+\ell \leq L\}$$

by the corresponding lower index set. In Step 3, each spatial block W_j is paired with the largest active temporal level allowed by that lower set. The same blockwise projection estimate then gives the identical lower bound c_T. This is the missing justification behind the "without loss of generality" reduction in the printed proof.